Formally, when β is small,

$\Rightarrow$ Maxwell - Boltzmann statistics

This happen when $e^{-\beta/\mu} \gg 1$

Then $\langle N \rangle \sim e^{-\beta(\epsilon_i - \mu)}$, which can happen only if $\mu < 0$

Recall, for classical gas:

$$-\beta\mu = \ln\left(\frac{V}{N\lambda^3}\right)$$

where

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$$

For $-\beta\mu \gg 1$, must have

$$\frac{V}{N} \gg \lambda^3$$

or

$$\frac{N}{V} \ll \frac{\sqrt{2\pi m k_B T}}{h}$$

Can happen only if T is large or number density is small a3/ For the opposite limit: $-\beta\mu \ll 1$

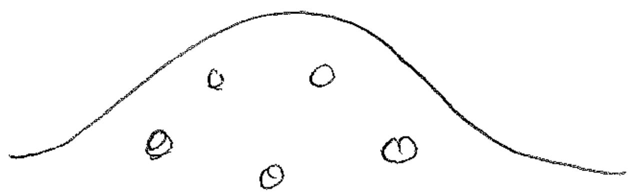
$$\lambda^3 \gg \frac{V}{N}$$

corresponds to degenerate case:

$\frac{V}{N} \rightarrow$ volume occupied by a single particle

$\left(\frac{V}{N}\right)^{1/3} \rightarrow$ size of a single particle

Thus, if size $\ll \lambda$ (or volume available to a particle) (mean distance between particles $\ll$ de Broglie)



particles effectively crowded
$\Rightarrow$ Pauli statistics important

Free Bosons

Similar procedures:

$$Z_G = \prod_{i=1}^{s} \left(\sum_{n=0}^{\infty} e^{-\beta n(\epsilon_i - \mu)} \right) =$$

$$= \prod_{i=1}^{s} \frac{1}{1 - e^{-\beta(\epsilon_i - \mu)}}$$

Similar manipulations lead to

$$\boxed{\langle n_i \rangle = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}}$$ ← mean occup. nu. of single particle state

How can this make sense?

Since $(\langle n_i \rangle) > 0$ always, we must have

$\{\mu < \epsilon_i\}$ for all ϵ_i

When T is such that

$$\mu(T_0) = \epsilon_0$$

which is lowest possible value, for which $\langle n_0 \rangle \longrightarrow \infty$

$\Longrightarrow$ BEC (i.e. BE condensate)

Need to examine in detail when $e^{\beta(\epsilon_i - \mu)} \gg 1$ for all i,

$$\langle n_i \rangle \sim e^{-\beta(\epsilon_i - \mu)}$$ ← MB statistics

This requires $e^{-\beta\mu} \gg 1$ or $\lambda \ll (V/N)^{1/3}$

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Fluctuations

Define an auxilliary function

$$\tilde{Z} = \sum_{\{n_i\}} e^{-\beta \sum n_i (\epsilon_i - \mu)}$$

Then
$$\langle n_i \rangle = \frac{1}{\beta} \frac{\partial \ln \tilde{Z}}{\partial \mu_i}$$

$$\langle n_i^2 \rangle = \frac{1}{\beta^2} \frac{1}{\tilde{Z}^2} \frac{\partial^2 \tilde{Z}}{\partial \mu_i^2}$$

May calculate and then set all $\mu_i = \mu$ for all i

$$\langle n_i^2 \rangle - \langle n_i \rangle^2 = \langle (\Delta n_i)^2 \rangle = \frac{1}{\beta^2} \left(\frac{\partial^2}{\partial \mu_i^2} \ln \tilde{Z}_i \right)_{\mu_i = \mu} =$$

$$= \frac{1}{\beta} \left(\frac{\partial}{\partial \mu_i} \langle n_i \rangle \right)_{\mu_i = \mu}$$

This is the same as

$$\langle (\Delta n_i)^2 \rangle = -\frac{1}{\beta} \frac{\partial}{\partial \epsilon_i} \langle n_i \rangle$$

$$\boxed{\frac{\langle (\Delta n_i)^2 \rangle}{\langle n_i \rangle^2} = \frac{1}{\beta} \frac{\partial}{\partial \epsilon_i} \left(e^{\beta(\epsilon_i - \mu)} \pm a \right)}$$

$$\langle n_i \rangle = \left[e^{\beta(\epsilon_i - \mu)} \pm a \right]^{-1} \text{ and } \left(\frac{1}{\beta} \frac{\partial}{\partial \epsilon_i} \frac{1}{\langle n_i \rangle} \right)$$

Thus $$\frac{\langle(\Delta n_i)^2\rangle}{\langle n_i\rangle^2} = e^{\beta(\epsilon_i-\mu)} = \frac{1}{\langle n_i\rangle} - a$$

$$\begin{cases} a=0 & \text{MB} \\ =1 & \text{FD} \\ =-1 & \text{BE} \end{cases}$$

At low T:

FD : for all $\epsilon_i < \mu$ $\quad \langle n_i\rangle \to 1$

$$\Rightarrow \frac{(\Delta n_i)^2}{\langle n_i\rangle^2} \Rightarrow 0$$

BE: $$\Rightarrow \frac{\langle(\Delta n_i)^2\rangle}{\langle n_i\rangle^2} = \frac{1}{\langle n_i\rangle} + 1$$

Does not vanish even at low T!

Probability of occupation

For greater understanding of statistics of occupation numbers:

$P_i(n)$ - probability that there are n particles in state i (of energy ϵ)

FD: $n = 0, 1$

$$\langle n_i \rangle = \sum_{n=0}^{1} n P(n) = P_i(1)$$

Thus,
$$\begin{cases} P_i(1) = \langle n_i \rangle \\ P_i(0) = 1 - \langle n_i \rangle \end{cases}$$

BE: $$Z_G = \prod_i \left(\sum_n e^{-n\beta(\epsilon_i - \mu)} \right)$$

Thus, $P_i(n) \propto e^{-n\beta(\epsilon_i - \mu)}$

Since $\sum P_i(n) = 1$

Normalized probability is $\rightarrow$ BE

$$P_i(n) = e^{-\beta(\epsilon_i - \mu)n} \left[1 - e^{-\beta(\epsilon_i - \mu)} \right]$$

Use $$\langle n_i \rangle = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}$$

$$\Rightarrow e^{\beta(\epsilon_i - \mu)} = 1 + \frac{1}{\langle n_i \rangle}$$

Thus, $$P_i(n) = \left(\frac{\langle n_i \rangle}{\langle n_i \rangle + 1} \right)^n \left(1 - \frac{\langle n_i \rangle}{\langle n_i \rangle + 1} \right) = \frac{\langle n_i \rangle^n}{(\langle n_i \rangle + 1)^{n+1}}$$

$$\Rightarrow \frac{P_i(n+1)}{P_i(n)} = \frac{\langle n_i \rangle}{\langle n_i \rangle + 1} \leftarrow \text{independent of } n$$

Thus, bosons tend to bundle together.

Contrast with MB:

$$P_i(n) = \frac{A}{n!} e^{-\beta(\epsilon_i - \mu)n}$$

Normalize: $A e^{-\beta(\epsilon_i - \mu)} = 1 \Rightarrow A = e^{\beta(\epsilon_i - \mu)}$

Since $\langle n_i \rangle = e^{-\beta(\epsilon_i - \mu)}$

$$P_i(n) = \frac{[e^{\beta(\epsilon - \mu)}]^n / n!}{\exp(e^{\beta(\epsilon - \mu)})} = \boxed{P_i(n) = \frac{1}{n!} \langle n_i \rangle^n e^{-\langle n_i \rangle}}$$

Poisson distribution (mean square deviation of variable = mean value itself)

$$\langle (\Delta n_i)^2 \rangle = \langle n_i \rangle$$

and $$\frac{P_i(n)}{P_i(n-1)} = \frac{(n-1)!}{n!} \frac{\langle n_i \rangle^n}{\langle n_i \rangle^{n-1}} = \frac{\langle n_i \rangle}{n-1}$$

This decreases with n

$\Rightarrow$ "normal" statistical behavior