Binomial and Poisson

Consider an event which has a probability $\mathcal{P}$ of occuring.

Question: what is the probability that in N trials the event will occur n times?

$\Rightarrow$ number of ways to pick n out of N integers

$$\frac{N!}{n!(N-n)!} = \binom{N}{n}$$

Hence, $$\boxed{W(n) = \binom{N}{n} \mathcal{P}^n (1-\mathcal{P})^n}$$

Clearly $$\sum_{n=0}^{N} W(n) = [\mathcal{P} + (1-\mathcal{P})]^N = 1$$

when $N \gg 1$, $\mathcal{P} \ll 1$, $n \ll N$

$$\Rightarrow \binom{N}{n} \rightarrow \frac{N^n e^{-N}}{n!\, N^{N-1} e^{-(N-n)}} = \frac{1}{n!} N^n e^{-n}$$

Note, Stirling: $N! = N^n e^{-N}$,
$(N-n)! = N^{N-n} e^{-(N-n)}$

So, $$\frac{N!}{n!(N-n)!} = \frac{1}{n!} N^n e^{-n}$$

$$(1-P)^{N-n} = e^{(N-n)\ln(1-P)} = \left[e^{\ln(1-P)}\right]^{N-n}$$

$$= e^{-P(N-n)}$$

$$P \ll 1 \leftarrow \ln(1-P) \approx (-P) + O(P^2)$$

Thus,

$$W(n) = \frac{1}{n!} N^n e^{-n} P^n e^{-P(N-n)} = \frac{1}{n!}(NP)^n e^{-NP} e^{-n(1-P)}$$

$$e^{-n(1-P)} \approx 1 \leftarrow [\underbrace{e^{1-P}}_{\sim 1}]^{-n} \leftarrow \begin{array}{l} P \ll 1 \\ n \ll N \end{array}$$

$$\Rightarrow W(n) \to \frac{1}{n!}(NP)^n e^{-NP} \quad \text{Poisson distrib.}$$

However, $NP = \langle n \rangle \leftarrow$ mean number of events

$$W(n) = \frac{1}{n!} \langle n \rangle^n e^{-\langle n \rangle}$$

$\Rightarrow$ just as MB!

(see p. 98)

General Relations of EOS

For both FD and BE

$$\ln Z_G = \frac{1}{a}\sum_i \ln\left[1 + a e^{-\beta(\epsilon_i - \mu)}\right]$$

$$a = \begin{cases} +1 & \text{FD} \\ -1 & \text{BE} \end{cases}$$

$\boxed{\text{Fugacity: } \quad z \equiv e^{\beta\mu}}$

For large volume:

$$\ln Z_G = \frac{4\pi V}{a h^3}\int_0^\infty dp\, p^2 \ln\left(1 + a z e^{-\beta\epsilon}\right) =$$

$$= \frac{4\pi V}{a h^3}\left\{\underbrace{\left[\frac{p^3}{3}\ln\left(1 + a z e^{-\beta\epsilon}\right)\right]_0^\infty}_{I_1} - \right.$$

$I_2 \Longrightarrow$

$$\left. - \int_0^\infty dp\, \frac{p^3}{3}\left[\frac{d}{dp}\ln\left(1 + a z e^{-\beta\epsilon}\right)\right]\right\}$$

Assume $\epsilon(p) = \alpha p^s \qquad s \geq 1$

Then, $\qquad \ln\left(1 + a z e^{-\beta\epsilon}\right) \sim a z e^{-\beta\alpha p^s}$

I_1 behavior for $p \to \infty$, $p \to 0$: $\left\{ \begin{array}{l} p^3 e^{-\alpha p^3} \to 0 \text{ as } p \to \infty \text{ if } s \geq 1 \\ \text{and } I_1 \to 0 \text{ for } p \to 0 \end{array} \right.$

In I_2: $\displaystyle \frac{d}{dp}\ln\left(1 + a z e^{-\beta\epsilon}\right) = -\frac{\beta a z \frac{d\epsilon}{dp} e^{-\beta\epsilon}}{1 + a z e^{-\beta\epsilon}} =$

$$= -\frac{\beta}{\frac{1}{az}e^{\beta\epsilon} + 1}\,\frac{d\epsilon}{dp}.$$

02) Rewriting I_2:
$$\ln Z_G = \frac{4\pi V}{h^3}\frac{\beta}{3}\int_0^\infty dp\, p^3 \frac{d\epsilon}{dp}\frac{1}{\frac{1}{z}e^{\beta\epsilon}+a}$$

However, the system is extensive. (for Ω_G - Grand Pot.)

Gibbs-Duhem:
$$\Rightarrow \Omega_G = -\frac{1}{\beta}\underbrace{\ln Z_G = -PV}$$

$$\boxed{P = \frac{4\pi}{3h^3}\int_0^\infty dp\, p^3 \frac{d\epsilon}{dp}\frac{1}{\frac{1}{z}e^{\beta\epsilon}+a}} \qquad (*)$$

Note, occupation number is:
$$\frac{1}{\frac{1}{z}e^{\beta\epsilon}+a} = \langle n\rangle \rightarrow N = \int \langle n\rangle \frac{V d^3p}{h^3} = \frac{4\pi V}{h^3}\int_0^\infty \frac{1}{\frac{1}{z}e^{\beta\epsilon}+a}p^3 dp$$

$\Uparrow$ eq(**)

Thus, $P = \frac{1}{3V}\sum_i p\frac{d\epsilon}{dp}n_i$

$$P = \frac{1}{3V}\int_0 d^3p\, p\frac{d\epsilon}{dp}n \Rightarrow \boxed{P = \frac{N}{3V}\left\langle p\frac{d\epsilon}{dp}\right\rangle} \Leftarrow \text{from eqs } (*) \,\&\, (**)$$

For $\epsilon = \alpha p^s$

$$\Downarrow$$

$$P = \frac{Ns}{3V}\alpha\int d^3p\, p p^{s-1} n$$

$$\boxed{P = \frac{s}{3}\left\langle\frac{E}{V}\right\rangle} \Leftarrow \frac{Ns}{3V}\int d^3p\, \epsilon n$$

where $\epsilon N = E$

$s = 1, 2$ is easy to recognize!

Fermi Gas

Non-relativistic particles;

(*) $$\ln Z_G = \frac{4\pi V}{h^3} \int_0^\infty dp\, p^2 \ln\left(1 + z e^{-\beta p^2/2m}\right)$$

Change of variables to

$$\epsilon = \frac{p^2}{2m} \qquad d\epsilon = \frac{p\,dp}{m}$$

$$p^2 dp = (2m\epsilon)\frac{m\,d\epsilon}{p} = m\,d\epsilon\sqrt{2m\epsilon}$$

Thus,

$$\boxed{\ln Z_G = \frac{4\pi V}{h^3} m^{3/2}\sqrt{2}\int_0^\infty d\epsilon\, \epsilon^{1/2}\ln\left(1 + z e^{-\beta\epsilon}\right)}$$

Define $f \equiv \frac{4\pi}{h^3} m^{3/2}\sqrt{2}$

$\alpha = -\beta\mu$, i.e., $z = e^{-\alpha} = e^{\beta\mu}$ $(0 \le z < \infty)$

Then, $\langle N\rangle = -\left(\frac{\partial \ln Z_G}{\partial \alpha}\right)$; $\langle E\rangle = -\left(\frac{\partial \ln Z_G}{\partial \beta}\right)$

$$\boxed{\begin{aligned} \langle N\rangle &= Vf\int_0^\infty d\epsilon\, \epsilon^{1/2}\frac{1}{z^{-1}e^{\beta\epsilon}+1} \\ \langle E\rangle &= fV\int_0^\infty d\epsilon\, \epsilon^{3/2}\frac{1}{z^{-1}e^{\beta\epsilon}+1} \end{aligned}}$$

Want to evaluate this for large values of z

Recall Gibbs-Duhem relation:

$$PV = -\Omega_G = T \ln Z_G$$

Thus, $$\frac{PV}{T} = \frac{4\pi V}{h^3} m^{3/2}\sqrt{2}\int_0^\infty d\epsilon\, \epsilon^{1/2} \ln\left(1+ze^{-\beta\epsilon}\right)$$

Define $\beta\epsilon \equiv x$

$$(*)\quad \frac{PV}{T} = \frac{4\pi V}{h^3} m^{3/2}\sqrt{2}\frac{1}{\beta^{3/2}}\int_0^\infty dx\, x^{1/2} \ln\left(1+ze^{-x}\right)$$

Note, coefficient is ($k_B = 1$)

$$\frac{(2\pi m T)^{3/2}}{h^3}\frac{2}{\sqrt{\pi}} = \frac{1}{\lambda^3}\frac{2}{\sqrt{\pi}}$$

$$\Rightarrow \frac{P}{T} = \frac{1}{\lambda^3}\frac{2}{\sqrt{\pi}}\int_0^\infty dx\, x^{1/2}\ln\left(1+ze^{-x}\right)$$

Integrate by parts:

$$\left[\frac{2}{3}x^{3/2}\ln\left(1+ze^{-x}\right)\right]_0^\infty - \frac{2}{3}\int_0^\infty x^{3/2}\frac{d}{dx}\ln\left(1+ze^{-x}\right) =$$

$$= \left[\frac{2}{3}x^{3/2}\ln\left(1+ze^{-x}\right)\right]_0^\infty + \frac{2}{3}\int_0^\infty dx\, x^{3/2}\frac{ze^{-x}}{1+ze^{-x}}$$

100 First term vanishes at both ends;

So,
$$\frac{P}{T} = \frac{1}{\lambda^3}\,\frac{2}{\sqrt{\pi}}\,\frac{2}{3}\int_0^\infty dx\,\frac{x^{5/2-1}}{\frac{1}{z}e^x+1}$$

$$(*)\quad \boxed{\frac{P}{T} = \frac{1}{\lambda^3}\,\frac{1}{\Gamma(5/2)}\int_0^\infty dx\,\frac{x^{5/2-1}}{\frac{1}{z}e^x+1} = \frac{1}{\lambda^3} f_{5/2}(z)}$$

where we define
$$\boxed{f_\nu(z) = \frac{1}{\Gamma(\nu)}\int_0^\infty dx\,\frac{x^{\nu-1}}{\frac{1}{z}e^x+1}}$$

Similarly, $\boxed{\dfrac{\langle N\rangle}{V} = \dfrac{1}{\lambda^3} f_{3/2}(z)}$

while, $\boxed{\dfrac{\langle E\rangle}{V} = \dfrac{3T}{2}\,\dfrac{1}{\lambda^3}\, f_{5/2}(z)}$

$$\frac{2}{3}\,\frac{\langle E\rangle}{V} = T\,\underbrace{\frac{1}{\lambda^3} f_{5/2}(z)}_{P}$$

(see above, eq. *)