Solutions to problem set # 1 PHY632

Solutions

(1) (a) From the equation of motion:

$$m\dot{v} + f v = 0, \quad (f \text{ - coeff of friction})$$

we find

$$\begin{cases} v = v_0 e^{-t/\tau} \\ q = q_0 + v_0\tau(1 - e^{-t/\tau}) \end{cases} \quad \text{where } \tau = \frac{m}{f}.$$

So, $p + v\tau = q_0 + v_0\tau \Rightarrow$ phase space trajectories on a plane (q, v) form a family of lines inclined by $-1/\tau$.

If $f = 0$ (no friction), $1/\tau \to 0$ and the trajectories are parallel to the coordinate axis, as $v = \text{const}$.

The Jacobian is

$$D \equiv \frac{\partial(q, v)}{\partial(q_0, v_0)} = e^{-t/\tau}$$

$\Rightarrow$ The phase space volume decreases with time.

(b) Solving the equation of motion

$$\ddot{q} + \frac{1}{\tau}\dot{q} + \omega_0^2 q = 0$$

we find, for $\frac{1}{\tau} \ll \omega_0$, that

$$q^2+\frac{v^2}{\omega^2}=\left(q_0^2+\frac{v_0^2}{\omega^2}\right)e^{-t/\tau}.$$

So, the point moves along the ellipse with the semi-major axis decreasing with time $\propto e^{-t/2\tau}$. This means that the point in the phase space moves along the elliptical spiral around the $(0,0)$ point.

The Jacobian is

$$D=e^{-t/\tau}$$

and the phase space volume decreases with time.

(2) After the collision, the coordinates are q_i' and p_i' $(i=1,2)$, while before the collision q_i, p_i.

Momentum conservation:

$$p_1'+p_2'=p_1+p_2$$

Energy conservation:

$$\frac{p_1'^2}{2m_1}+\frac{p_2'^2}{2m_2}=\frac{p_1^2}{2m_1}+\frac{p_2^2}{2m_2}$$

Hence, $\begin{cases} P_1' = \dfrac{m_1 - m_2}{m_1 + m_2} P_1 + \dfrac{2m_1}{m_1+m_2} P_2 \\ P_2' = -\dfrac{m_1 - m_2}{m_1 + m_2} P_2 + \dfrac{2m_2}{m_1+m_2} P_1 \end{cases}$

The Jacobian of coordinate transformation is.

$$D = \frac{\partial(q_1', q_2', P_1', P_2')}{\partial(q_1, q_2; P_1, P_2)}.$$

Because, $\dfrac{\partial P_i'}{\partial q_j} = 0$; $\dfrac{\partial q_i'}{\partial q_j} = \delta_{ij}$

$$\Longrightarrow \quad D = \frac{\partial(P_1', P_2')}{\partial(P_1, P_2)}.$$

So, $|D| = 1 \Rightarrow$ phase space volume is conserved

(3) Find derivative:

$$\frac{d}{dt}\int_{(x)} F\{\rho(x,t)\}\, dx = \int_{(x)} F' \frac{\partial \rho}{\partial t}\, dx$$

where $F' = dF/d\rho$.

From the Liouville theorem,

$$\int_{(X)} F' \frac{\partial \rho}{\partial t} dx = \int_{(X)} F' [H, \rho]_{PB}\, dx =$$

$$= \int_{(X)} F' \sum_{k=1}^{3N} \left[\frac{\partial H}{\partial q_k} \frac{\partial \rho}{\partial p_k} - \frac{\partial H}{\partial p_k} \frac{\partial \rho}{\partial q_k} \right] dx =$$

$$= \sum_{k=1}^{3N} \int_{(X)} \left[\frac{\partial H}{\partial q_k} \frac{\partial F}{\partial p_k} - \frac{\partial H}{\partial p_k} \frac{\partial F}{\partial q_k} \right] dx .$$

Because $\frac{\partial H}{\partial q_k}$ is independent of p_k, and $\frac{\partial H}{\partial p_k}$ of q_k, and taking into account that $F \to 0$ when $\rho = 0$, and $\rho \to 0$ when $q_k \to \infty$, $p_k \to \infty$:

$$\begin{cases} \int_{-\infty}^{+\infty} \frac{\partial F}{\partial p_k} dp_k = F \Big|_{p_k=-\infty}^{p_k=+\infty} = 0 \\ \int_{-\infty}^{+\infty} \frac{\partial F}{\partial q_k} dq_k = F \Big|_{q_k=-\infty}^{q_k=+\infty} = 0; \end{cases}$$

So indeed, $\frac{d}{dt} \int_{(X)} F\{\rho(x,t)\}\, dx \equiv 0.$

4 (a) All spins down → max. possible energy $N\mu H$

All spins up → min. possible energy $-N\mu H$

spacing between energy levels: $2\mu H$

(b) The number of microstates:

$$\frac{N!}{N_{up}!\,N_d!}$$

These states correspond to energy level

$$E = -(N_{up} - N_d)\,\mu H$$

(c) The # of configurations in this energy range is about the number of energy levels in the range times the number of configurations in a typical energy level. If N_{up} and N_d represent typical values for a level within this range, then

$$\Omega \sim \left(\frac{N!}{N_{up}!\,N_d!}\right)\left(\frac{\delta E}{2\mu H}\right)$$

To write this as a function of macroscopic quantities only → solve for N_{up} and N_d in terms of E, N and H. First:

$$\begin{cases} N_{up} + N_d = N \\ N_{up} - N_d = -\dfrac{E}{\mu H} \end{cases},$$

Solution: $N_{up} = \frac{1}{2}(N - E/\mu H)$; $N_d = \frac{1}{2}(N + E/\mu H)$

$$\Rightarrow \Omega(E, \delta E, H, N) \sim N!/\{[\tfrac{1}{2}(N+E/\mu H)]!\,[\tfrac{1}{2}(N-E/\mu H)]!\}\left(\frac{\delta E}{2\mu H}\right).$$